

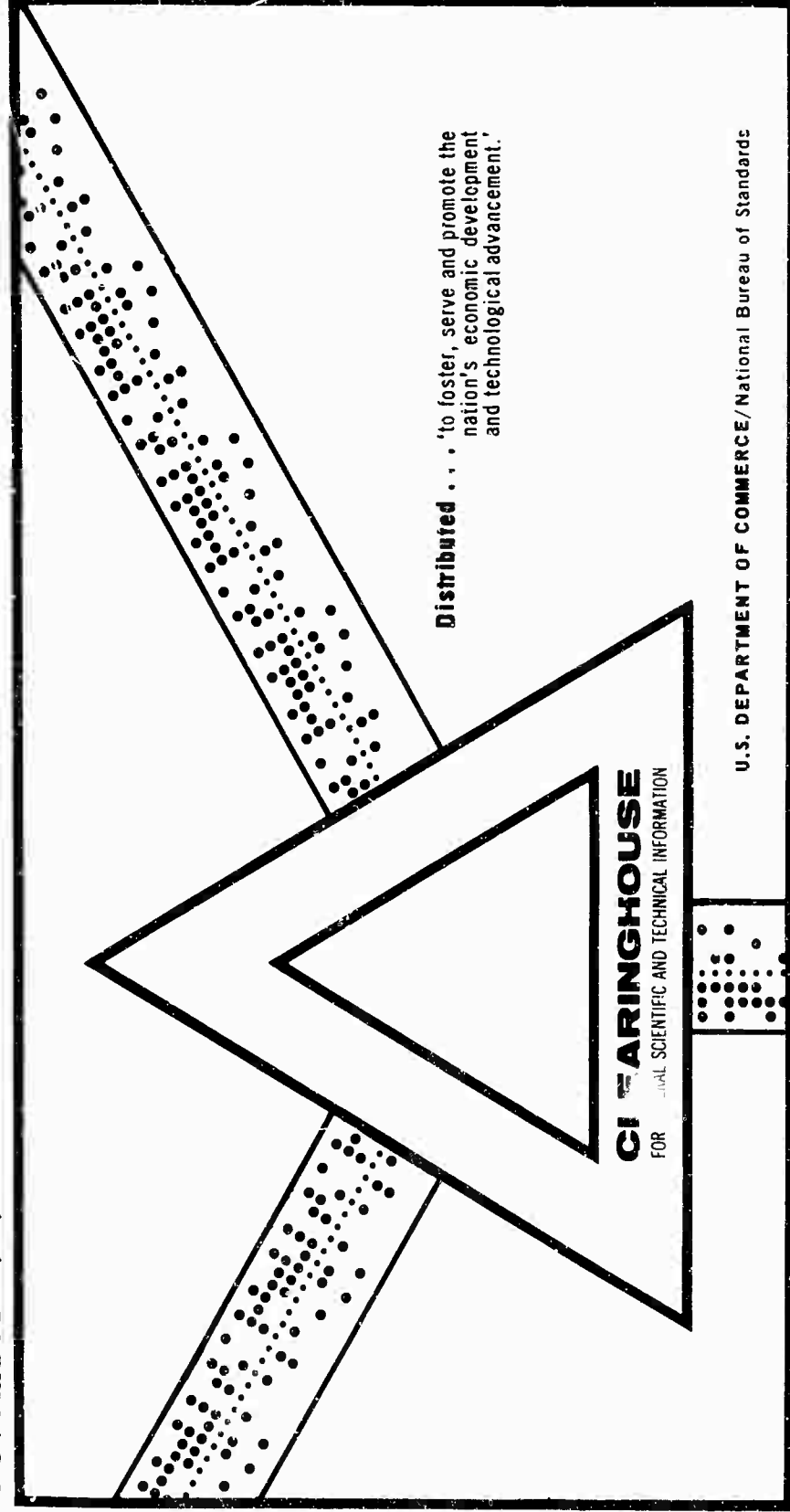
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THE AXISYMMETRIC TURBULENT BOUNDARY LAYER ON AN EXTREMELY  
LONG CYLINDER

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Navy Underwater Sound Laboratory  
New London, Connecticut

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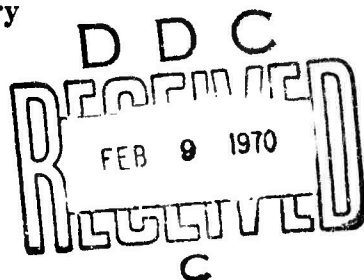


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### ABSTRACT

An analysis is presented which predicts the properties of an arbitrarily thick turbulent boundary layer in axial flow past a long cylinder. The study makes use of a modified form of the turbulent law-of-the-wall, which properly accounts for transverse curvature effects. Using this law, the theory which follows is then an exact solution to the axisymmetric equations of continuity and momentum in incompressible flow.

Numerical results are given to show the effect of curvature on the various boundary layer characteristics. Skin friction and drag coefficients can be increased greatly with increasing curvature while boundary layer thickness is decreased. When defined in their axisymmetric form, the displacement and momentum thickness are both decreased by curvature. The velocity profile is flattened greatly and the shape factor  $H = \delta^*/\theta$  approaches unity at large curvature. The failure of earlier power-law theories to make accurate predictions is shown to be due to their inadequate handling of the strong profile shape changes. Finally, the earlier concept that the curvature effect on skin friction could be correlated by the ratio  $(\theta/d)$  is shown to be invalid.

### ADMINISTRATIVE INFORMATION

Dr. White, a professor of Mechanical and Ocean Engineering at the University of Rhode Island, serves the Navy Underwater Sound Laboratory as a Consultant to the Special Developments Branch of the Submarine Sonar Division, Systems Department. This report is the result of a study performed for the Laboratory by Dr. White under a personal services contract. The Laboratory project number is 9-A-509-00-00, and the corresponding Navy subproject and task number is SF 35 452 007-01356. Dr. Henry P. Bakewell, Jr., Senior Project Engineer in the Special Developments Branch, provided the liaison for this investigation and was the Technical Reviewer for this report.

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## NOMENCLATURE

$a$	= cylinder radius
$B, k$	= law-of-the-wall constants, Eq. (2)
$C_f$	= local friction coefficient = $2\tau_w/\rho U^2$
$C_D$	= overall drag coefficient = $2(\text{Drag})/\rho U^2 2\pi aL$
$d$	= cylinder diameter = $2a$
$f$	= law-of-the-wall function, Eq. (12)
$H$	= shape factor = $\delta^*/\theta$
$L$	= cylinder length
$r, x$	= radial and axial coordinates
$R_x, R_L$	= local and overall Reynolds numbers
$u, v$	= axial and radial velocity components
$U$	= freestream velocity
$v^*$	= wall friction velocity = $\sqrt{\tau_w/\rho}$
$y$	= wall coordinate = $[r-a]$
$Y$	= modified law-of-the-wall variable, Eq. (16)
$\delta$	= boundary layer thickness, Fig. 1
$\delta^*, \theta$	= displacement and momentum thickness, Eqs. (24), (25)
$\rho$	= fluid density
$\mu$	= fluid absolute viscosity
$\nu$	= fluid kinematic viscosity = $\mu/\rho$
$\tau$	= boundary layer shear stress
$\lambda$	= dimensionless friction variable = $U/v^* = \sqrt{2/C_f}$

### Subscript:

$w$  = at the wall

### Superscript:

$+$  = law-of-the-wall variable, Eqs. (1), (7)

# THE AXISYMMETRIC TURBULENT BOUNDARY LAYER ON AN EXTREMELY LONG CYLINDER

## INTRODUCTION

The prediction of boundary layer properties in axial flow past long bodies of revolution is an important practical problem. Such bodies are common in both aerodynamics and hydrodynamics, and one wishes to know whether the transverse curvature has any important effect in estimating, say, skin friction or displacement thickness. If the flow is turbulent, the prediction of wall pressure fluctuations, along with the coupled dynamic response that such flow noise might cause to the body, is important. As a first step in such problems, it is the purpose of this report to analyze the axial flow turbulent boundary layer past an extremely long circular cylinder. The analysis is based upon a cylindrical version of the turbulent law-of-the-wall, first deduced by G. N. V. Rao.<sup>13</sup> The present study is limited to incompressible flow with constant properties.

## LITERATURE REVIEW

First let us remark that the case of laminar flow past a long cylinder has been dealt with very successfully. The laminar boundary layer equations achieve closure without any extra empiricism needed, and accurate analytical solutions have been given by Probst and Elliott<sup>1</sup> and by Glauert and Lighthill.<sup>2</sup> Other laminar flow theories, mostly introductory in nature, are given by Seban and Bond,<sup>3</sup> Kelly,<sup>4</sup> and Cooper and Tulin.<sup>5</sup> Very strong curvature effects occur in laminar flow. For example, at  $x/d = 100$ , the local skin friction is increased nearly 100 percent over a flat plate with the equivalent Reynolds number. A theory for supersonic laminar cylinder flow was given by Mark.<sup>6</sup>

As might be expected, the bulk of experimental data occur under turbulent flow conditions, which is usually the case also in practical applications. Probably the earliest data on axial cylinder flows are due to Kempf,<sup>7</sup> and subsequent experiments have been performed by Telfer,<sup>8</sup> Hughes,<sup>9</sup> Richmond,<sup>10</sup> Yu,<sup>11</sup> Yasuhara,<sup>12</sup> and Rao.<sup>13</sup> The data of Richmond<sup>10</sup> are the most extensive and encompass both incompressible and hypersonic turbulent flows. The experiment of Yasuhara<sup>12</sup> included laminar flow conditions and showed transition to turbulence at a Reynolds number of  $R_x \doteq 1.5 \times 10^6$ , not much different from typical smooth flat plate transition points.

The earliest theory for turbulent cylindrical flow is due to Millikan,<sup>14</sup> who established the proper form of the Karman integral relation for axisymmetric flow. Millikan's calculations, which use the familiar  $1/7$  power law for the velocity profile, do not agree with experiment, because the  $1/7$  power is not at all accurate when curvature effects exist. Further theories using power-law profiles were given by Landweber,<sup>15</sup> Eckert,<sup>16</sup> Karhan,<sup>17</sup> and Sakiadis.<sup>18</sup> These results do not agree with experiment either and generally predict curvature effects that are too small by an order of magnitude.

A rather different approach was taken by Ginevskii and Solodkin,<sup>19</sup> who adapted Prandtl's mixing length theory to the thick cylindrical boundary layer. Their theory, which is extremely complex algebraically, includes both concave and convex surfaces and pressure gradient and flow separation effects. Unfortunately, their mixing length parameters were taken from flat plate data and thus are not accurate under strong curvature conditions.

#### THE LAW-OF-THE-WALL FOR A CYLINDER

As is well known, the equivalent shear stress in turbulent flow is not simply related to viscosity and velocity gradient but includes a large fluctuating inertia term, the physics of which is not well defined even to the present day. However, in two-dimensional flow, under flat plate (zero pressure gradient) conditions, it is a fortunate circumstance that the local velocity  $u(x,y)$  may be related with great accuracy to local wall variables:  $u = \text{fcn}(y, \rho, \mu, \tau_w)$ . In dimensionless form this becomes the celebrated law-of-the-wall:

$$u^+ = u/v^* = \text{fcn}(y^+), \quad (1)$$

where  $y^+ = \rho v^* y / \mu$  and  $v^* = \sqrt{\tau_w / \rho}$ . Very close to the wall, viscous shear is dominant and a sublayer occurs, where  $u^+ = y^+$ . Away from the wall, turbulent shear predominates, resulting in a logarithmic region:

$$u^+ = \frac{1}{k} \ln(y^+) + B, \quad (2)$$

where  $k$  and  $B$  are empirical constants that hold accurately for all flat plate or modest pressure gradient flows. Coles<sup>20</sup> suggests ( $k = 0.41$ ,  $B = 4.9$ ), while Spalding<sup>21</sup> prefers the nearly equivalent values ( $k = 0.40$ ,  $B = 5.5$ ). The difference is not important.



Finally, very far from the wall, a wakelike flow (Coles<sup>20</sup>) occurs which is independent of viscosity. We shall not consider this wake in the present report, because numerous theories have shown that the outer wake, while very interesting physically, has a negligible numerical effect on boundary layer computations such as skin friction and displacement thickness.

The law-of-the-wall provides the necessary closure relation for the turbulent boundary layer, relating wall shear to  $u$  and  $y$ . No further assumptions are needed for incompressible flow, and Eq. (2) may be combined with the continuity and momentum relations to yield very accurate and simple flat plate turbulent calculations, as in the theory of Brand and Persen.<sup>22</sup>

For cylinder flow, the entire concept must be reevaluated. Here the cylinder radius becomes an additional parameter, and we may postulate that  $u = \text{fcn}(r, a, \rho, \mu, r_w)$ , leading to any of several possible dimensionless forms; for example,

$$\begin{aligned} u^+ &= \text{fcn}(a^+, r/a) & (\text{Rao}^{13}) \\ \text{or} \\ u^+ &= \text{fcn}(y^+, y/a) & (\text{Richmond}^{10}), \end{aligned} \tag{3}$$

where  $y$  is the wall coordinate such that  $r = y + a$ . See Fig. 1 for a definition sketch of terms. The formulation of Richmond above, the earlier of the two by ten years, was based upon the so-called "streamline hypothesis" of Coles.<sup>20</sup> Coles was the first to show that, in two-dimensional flow, lines of constant  $u^+$

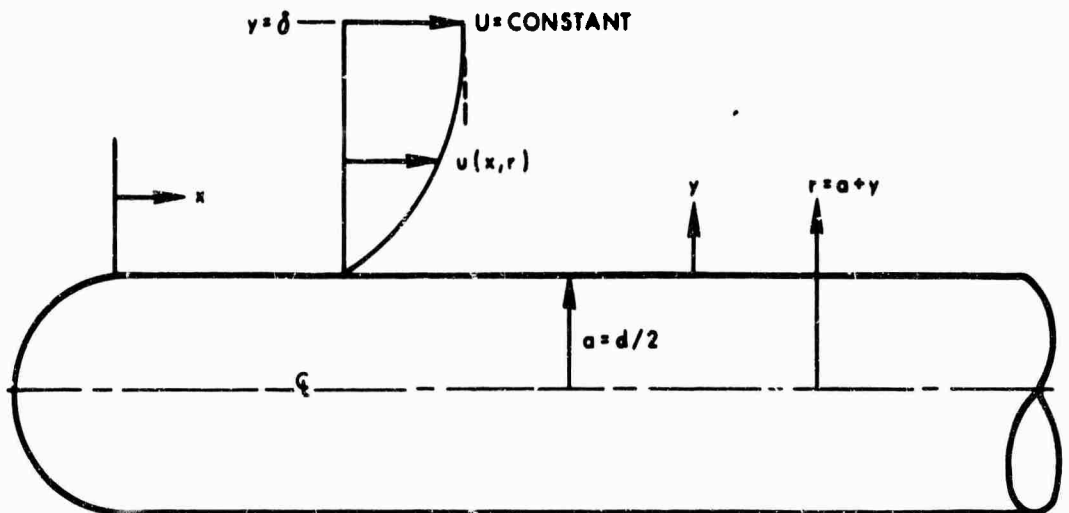


Fig. 1. Definition Sketch for Axial Cylinder Flow

are streamlines of the flow. Richmond<sup>10</sup> hypothesized that the same would be true in cylindrical flow, where the stream function varies as  $(r^2 - a^2)$ . If this were true,  $u^+$  would then be a function of only a single parameter  $[(a + y)^2 - a^2]$ , instead of  $a$  and  $y$  separately. Thus, Richmond proposed that

$$u^+ = \text{fcn}\left[v^*(r^2 - a^2)/2a\nu\right] = \text{fcn}\left[y^+(1 + y/2a)\right]. \quad (4)$$

This relation is also used to estimate  $v^*$  from the data. Upon plotting the results from various experiments in this manner — see Fig. 7 of Yasuhara<sup>12</sup> — the curves do indeed correlate very near the wall. Away from the wall, though, the curves begin to drop off significantly from the log law, Eq. (2). Furthermore, the deviations are a rather erratic function of the relative boundary layer thickness  $(\delta/a)$ . These facts tended to discourage further analytical work along law-of-the-wall lines, until, in 1967, Rao<sup>13</sup> pointed out the flaw in Richmond's approach.

The critical point made by Rao is that Richmond's proposal, Eq. (4), fails to describe the sublayer properly. Equation (4) leads to the sublayer proportionality

$$u^+ \doteq y^+(1 + y/2a) \quad (\text{Richmond}^{10}). \quad (5)$$

However, in the sublayer, it was shown by Rao that the true relation is quite different. Inertia is negligible in the sublayer, and hence the boundary layer momentum equation for zero pressure gradient reduces to

$$\frac{\partial}{\partial r}(r\tau) \doteq 0, \quad (6)$$

or

$$r\tau = a\tau_w = r\mu \frac{\partial u}{\partial r} \doteq \text{constant}.$$

Integrating this, we obtain the proper sublayer relation

$$u^+ = a^+ \ln(r/a), \quad (7)$$

where  $a^+ = v^*a/\nu$ . The power series approximation of the logarithm for small  $y$  leads to the relation

$$u^+ \doteq y^+(1 - y/2a) \quad (\text{Rao}^{13}). \quad (8)$$

The difference in sign of the parenthesis terms in Eqs. (5) and (8) is profound when the sublayer is thick, which of course it is whenever curvature is important. Thus, Richmond's expression fails just when we need it most: in estimating the wall shear. This is shown by solving Eqs. (5) and (7) for the wall shear:

$$\begin{aligned} \text{Richmond:} \quad \tau_w &= \mu u/y (1 + y/2a) \\ \text{Rao (exact):} \quad \tau_w &= \mu u/a \ln (1 + y/a). \end{aligned} \tag{9}$$

Thus, Richmond's expression underestimates the wall shear by the factor  $[(y/a) (1 + y/2a)/\ln (1 + y/a)]$ . The difference is negligible if  $(y/a)$  is very small (thin boundary layer), but in Richmond's thick-layer experiments, the measurement closest to the wall was at about  $y/a = 0.5$ . The error involved is the factor  $[0.5 (1.25)/\ln (1.5)] = 1.54$ ; the true shear was actually 54 percent larger than Richmond's tabulated values.

From his sublayer estimate, Eq. (7), Rao<sup>13</sup> then hypothesized that the cylindrical law-of-the-wall might actually satisfy the usual two-dimensional formulas, if  $y^+$  is replaced by the new sublayer variable  $Y = a^+ \ln (r/a)$ . For example, the logarithmic layer on a cylinder, using Spalding's constants, would be given by

$$u^+ = 2.5 \ln (Y) + 5.5, \quad Y = a^+ \ln (r/a). \tag{10}$$

Figure 2 shows all available data on thick cylindrical turbulent boundary layers, plotted in the suggested form  $u^+$  versus  $Y$ . It is seen that Rao's hypothesis is indeed the correct one. The agreement is good enough, considering the wide variety of relative curvatures, so that we may regard Fig. 2 as the correct closure relation needed to complete the analysis of cylindrical flows.

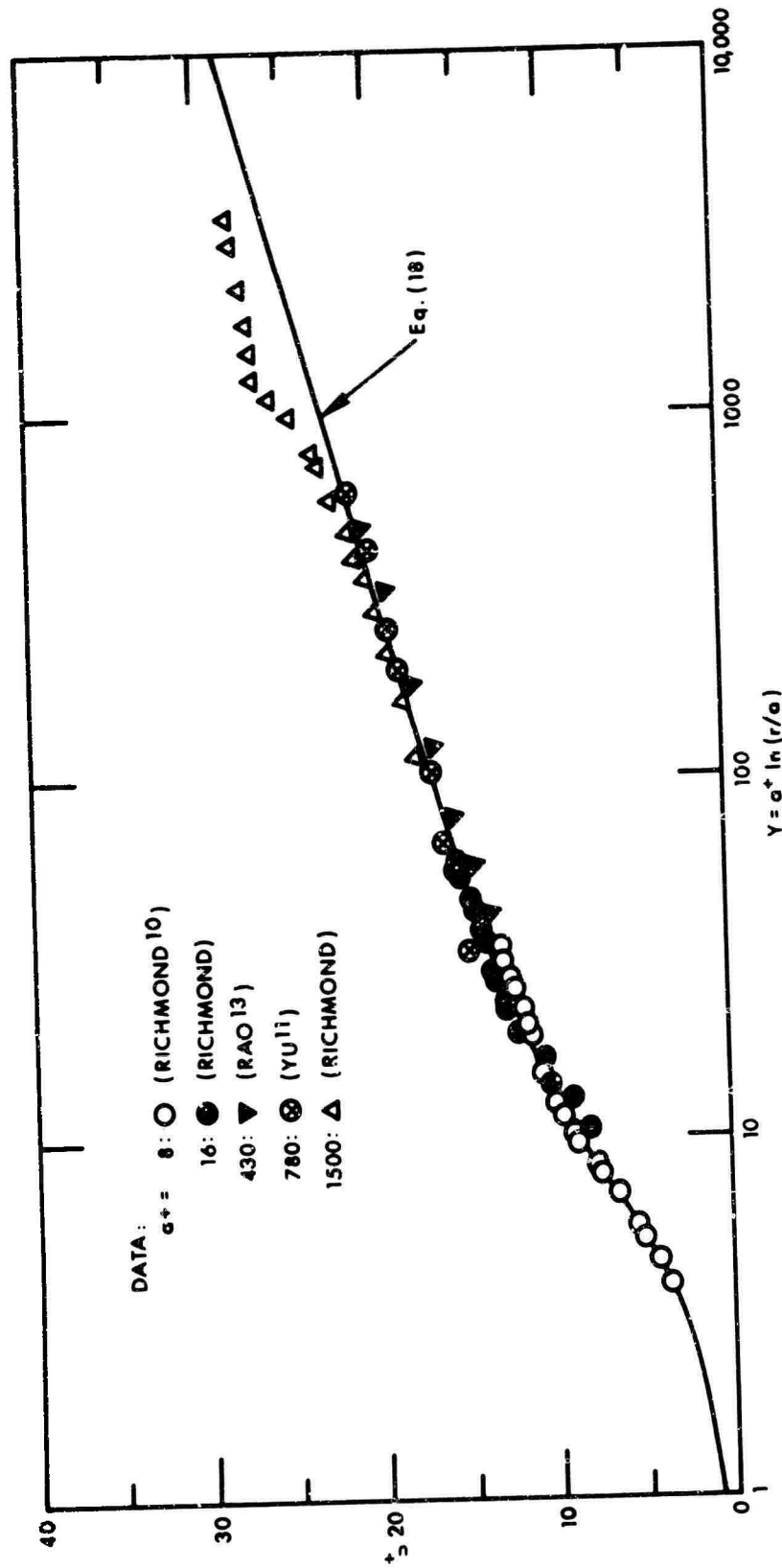


Fig. 2. The Modified Law-of-the-Wall for Axisymmetric Turbulent Flow

## THE THICK TURBULENT CYLINDRICAL BOUNDARY LAYER

### SKIN FRICTION AND DRAG

The modified law-of-the-wall from Fig. 2 is the missing link needed to complete the theory of turbulent flow past an extremely long cylinder. For incompressible axisymmetric flow with zero pressure gradient, the two basic relations are the equations of continuity and momentum:

$$\text{Continuity: } \frac{\partial(ru)}{\partial x} + \frac{\partial(rv)}{\partial y} = 0 \quad (11)$$

$$\text{Momentum: } ru \frac{\partial u}{\partial x} + rv \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial y} (r\tau).$$

If, now, the law-of-the-wall is given by the functional relation

$$u(x, y) = v^*(x) f(Y), \quad (12)$$

then Eqs. (11) and (12) may be solved directly for the wall shear. The velocity  $v$  may be eliminated through continuity:

$$rv = -\frac{\partial}{\partial x} \int_0^y ru \, dy = -\frac{\nu}{a} r^2 f \frac{\partial Y}{\partial x}. \quad (13)$$

Substitution of  $u$  and  $v$  from Eqs. (12) and (13) gives the following differential equation in the wall shear stress alone:

$$\rho r v^* \frac{dv^*}{dx} f^2 = \frac{\partial}{\partial y} (r\tau), \quad (14)$$

which we may integrate from the wall to the outer edge of the boundary layer:

$$\tau_w = -\mu \frac{dv^*}{dx} \int_0^{Y(\delta)} f^2 (r/a)^2 dY. \quad (15)$$

This relation is essentially identical to the flat plate skin friction expression derived by Brand and Persen,<sup>22</sup> except for the additional "curvature" term  $(r/a)^2 = \exp [2Y/a^+]$ . Clearly the wall shear is increased by curvature, the effect being negligible if  $r \doteq a$  (thin boundary layer).

Let us now define a dimensionless skin friction variable related to the usual coefficient  $C_f$ . Let

$$\lambda = U/v^* = \sqrt{2/C_f}, \quad (16)$$

where  $C_f = 2\tau_w/\rho U^2$ . Substitution into Eq. (15) gives a relation between the local Reynolds number and the skin friction coefficient, which is the central result of this report:

$$Ux/\nu = R_x = \int_0^\lambda G(\lambda) d\lambda, \quad G = \int_0^{Y(\lambda)} f^2 \exp [2Y/a^+] dY. \quad (17)$$

It merely remains to carry out the double integration indicated by Eq. (17), by using some suitably accurate representation of the modified law-of-the-wall,  $u^+ = f(Y)$ . The logarithmic law, Eq. (10), is reasonably accurate at large  $R_x$ , provided care is taken to eliminate the unwanted effects of the singularity at the wall. However, the sublayer becomes increasingly important with increased curvature. Hence, for maximum accuracy, we choose to work with the more cumbersome relation of Spalding,<sup>21</sup> which fits all three wall regions — sublayer, buffer layer, and log layer — very accurately:

$$Y \doteq u^+ + 0.1108 [e^z - 1 - z - z^2/2 - z^3/6 - z^4/24], \quad (18)$$

where  $z = ku^+ \doteq 0.4 u^+$ .

The constant 0.1108 corresponds to the choice  $B = 5.5$  in Eq. (2). Equation (18) is plotted for comparison in Fig. 2, and the agreement in all regions is seen to be good. Note that we require the modified variable  $Y = a^+ \ln(1 + y/a)$ , not  $y^+$ . Since Eq. (18) is a bulky and implicit form of  $f(Y)$ , its use in Eq. (17) makes numerical integration a necessity. The author's numerical results, which were obtained by use of a fourth-order Runge-Kutta procedure, are shown in Fig. 3 as a plot of  $C_f = 2/\lambda^2$  versus  $R_x$ , with the position  $(x/d)$  on the cylinder as a parameter. The original parameter,  $a^+$ , is eliminated through the relation  $x/d = R_x/2a^+\lambda$ . Note that realistic values of curvature ( $x/d = 10^5$ ) may increase the local skin friction by a factor of ten or more; this result is made possible by the profound changes in the velocity profile implied by the modified law-of-the-wall, Eq. (12).

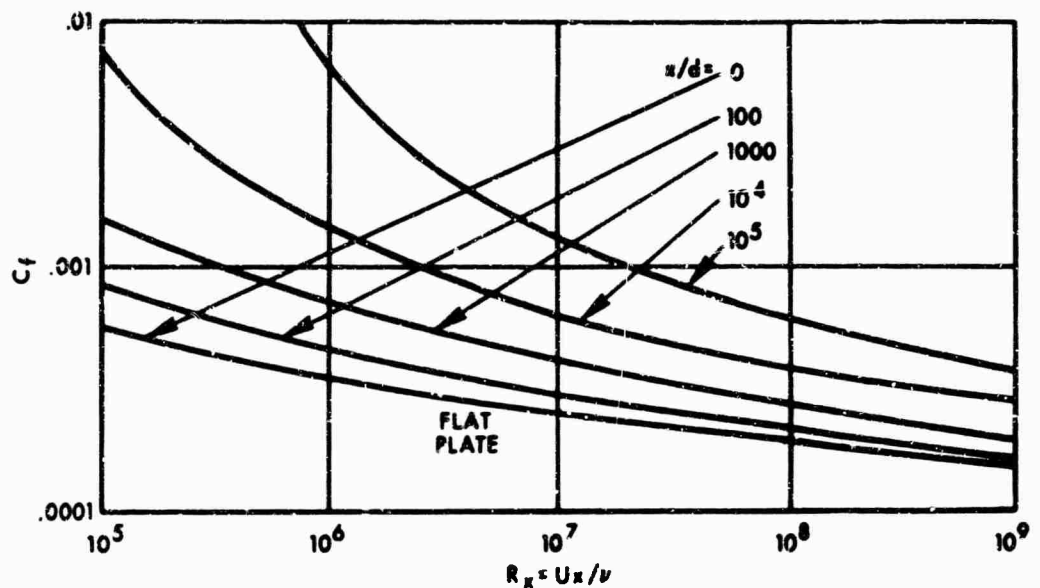


Fig. 3. Turbulent Skin Friction on a Long Cylinder, from Eqs. (17) and (18)

The total drag on the cylinder is evaluated by integrating the shear stress, and the drag coefficient  $C_D$  is defined with respect to the wetted area:

$$\text{Drag} = \int_0^L \tau_w 2\pi a \, dx ,$$

and (19)

$$C_D = \frac{2 \text{ Drag}}{\rho U^2 2\pi a L} .$$

Drag calculations were performed in two ways and were checked against each other to four significant figures. First, from the definition of  $C_f$ , Eq. (19) may be rewritten as

$$C_D = \int_0^1 C_f \, d(x/L) . \quad (20)$$

Second, if the leading edge is a uniform flow  $u = U$ , the drag may be calculated by a momentum integral across the trailing edge:

$$C_D = (2/L) \int_0^\delta [u/U(1 - u/U)(1 + y/a)]_{x=L} dy, \quad (21)$$

where the velocity profile is found from the modified law-of-the-wall, Eq. (18), and the known value of  $C_f$  at the trailing edge. The two approaches should give identical results. The numerical drag results are plotted in Fig. 4 as a

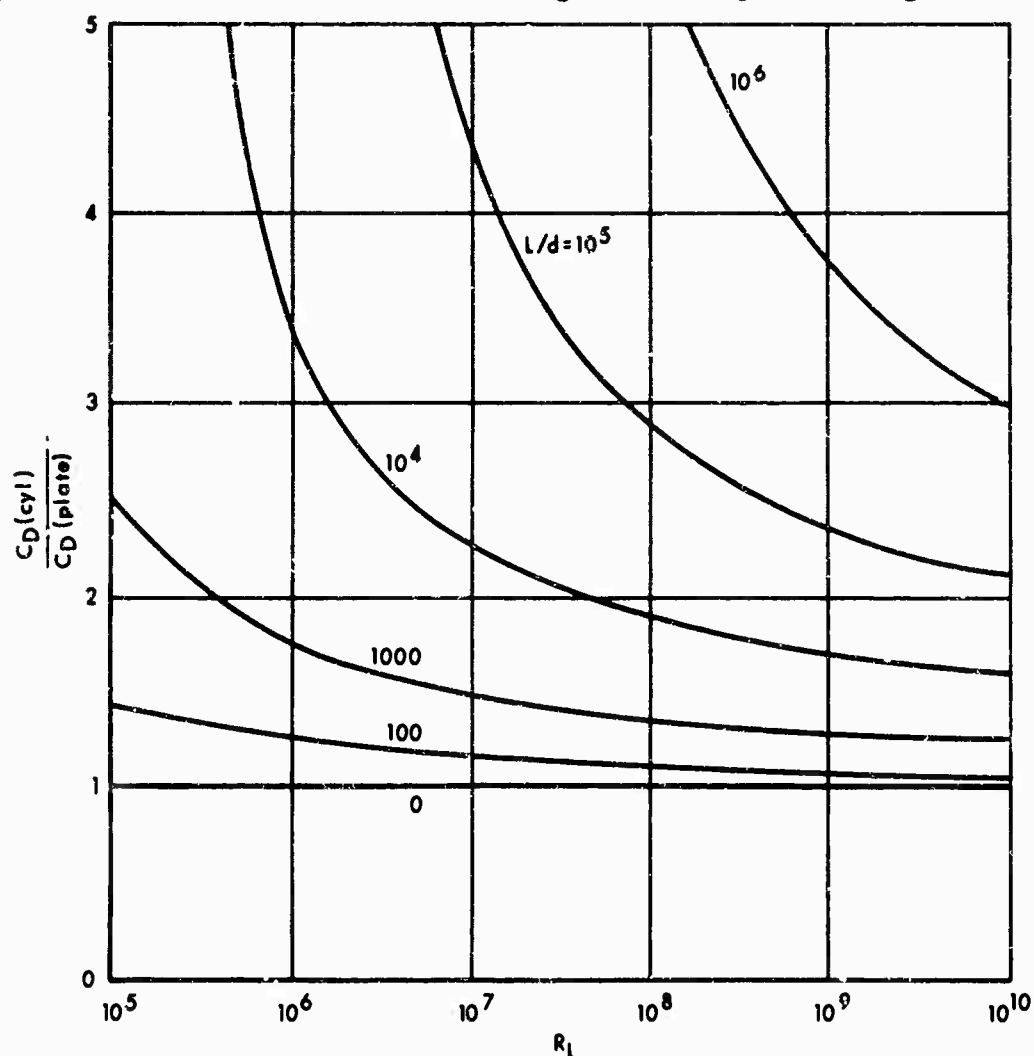


Fig. 4. Drag Coefficient of a Long Cylinder Compared with That of a Flat Plate at the Same Reynolds Number



ratio of  $C_D$  on the cylinder to the equivalent value for a flat plate, which is correlated accurately by the well known formula (see, for example, Schlichting,<sup>23</sup> Eq. (21.16)).

$$C_{D(\text{flat plate})} \doteq 0.455 / [\log_{10} R_L]^{2.58}. \quad (22)$$

Figure 4 shows that the drag of a long cylinder may be up to five times the equivalent plate drag. Drag ratios higher than five are probably not realistic, because the viscous sublayer then extends almost entirely across the boundary layer and raises stability questions about the assumption of turbulence.

#### VELOCITY PROFILES AND INTEGRAL THICKNESSES

Although the drag and skin friction of the cylinder are the primary results of this report, certain applications — e.g., wall pressure fluctuations — are related to the shape and thickness of the velocity profiles.

Once  $C_f = 2/\lambda^2$  is known, the boundary layer thickness is specified by the modified law-of-the-wall:  $\lambda = f[Y(\delta)]$ . That is,  $\lambda$  determined  $Y(\delta)$  and then  $\delta$  is calculated from

$$\delta/a = \exp [Y(\delta)/a^+] - 1. \quad (23)$$

Similarly, the law-of-the-wall determines  $u/U = u^+/\lambda$ , so that the profile shape is a function of two parameters:  $(C_f, u^+)$  or, better,  $(R_x, x/d)$ . Curvature causes the profile to be much flatter, and typical shapes (for  $R_x = 10^7$ ) are shown in Fig. 5. Also shown are values of the exponent  $n$  that make the curves fit approximately to the familiar power law  $(u/U) \doteq (y/\delta)^n$ . We see that the curvature changes the power in this case from about 1/11th to about 1/24th. This is one reason for the failure of the earlier power-law theories<sup>14-18</sup> to make accurate predictions; another reason is simply that the power law is not really a good fit anyway, especially in the vital sublayer region which curvature emphasizes.

The momentum and displacement thicknesses can also be calculated from the law-of-the-wall. Kelly<sup>4</sup> was the first to point out that the definition of these thicknesses should change to account for the cylindrical geometry. Thus,

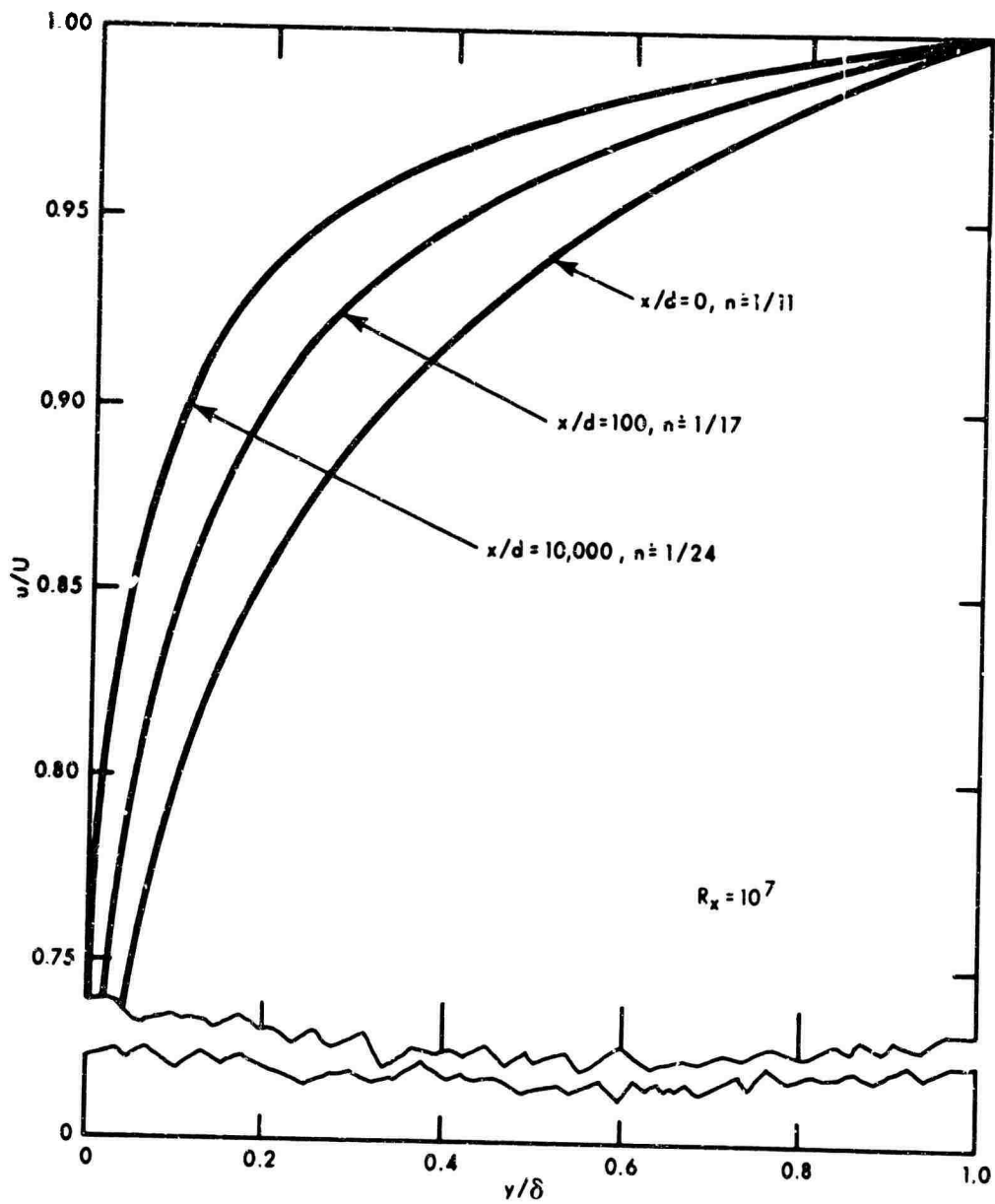


Fig. 5. Typical Effects of Curvature on Turbulent Velocity Profiles on a Long Cylinder, from Eqs. (18) and (19)

the displacement thickness on a cylinder is given by the formula

$$(\delta^* + a)^2 - a^2 = \int_a^{a+\delta} [1 - u/U] 2r dr, \quad (24)$$

and the momentum thickness is given by

$$(\theta + a)^2 - a^2 = \int_a^{a+\delta} (u/U) [1 - u/U] 2r dr. \quad (25)$$

No other definition is proper because it will not properly model the cylindrical mass flow relations along the cylinder. Values of  $\delta$  and  $\delta^*$  calculated from these relations are shown in Fig. 6 for various Reynolds numbers and curvatures.

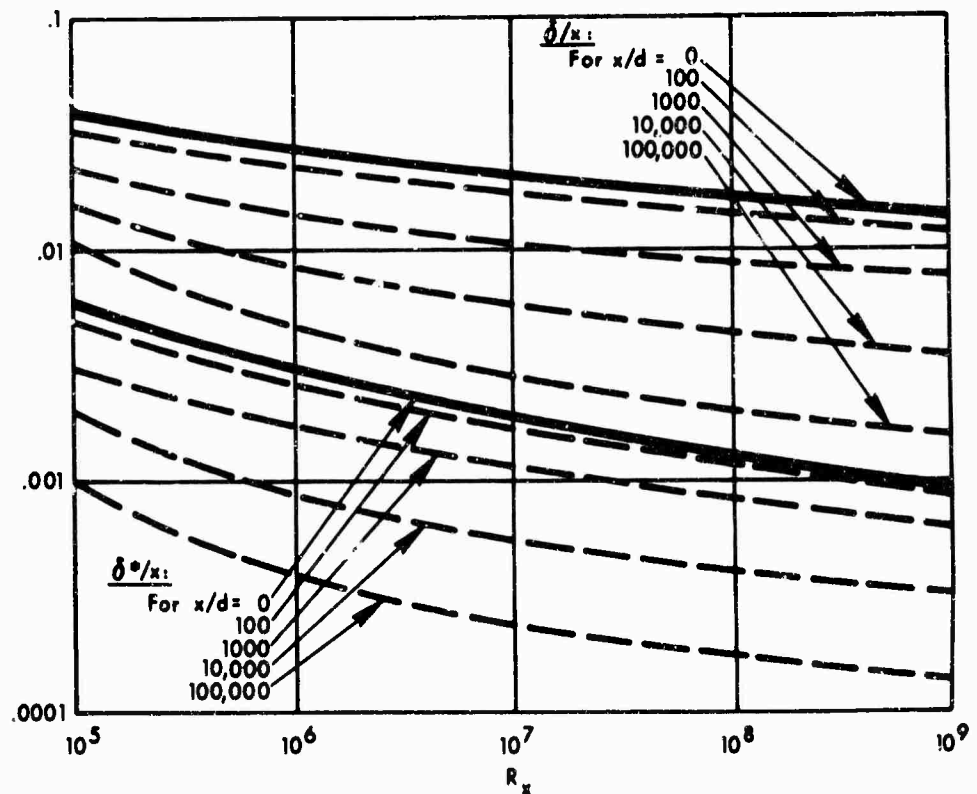


Fig. 6. Boundary Layer and Displacement Thicknesses, from Eqs. (23) and (24)

We see that curvature reduces both  $\delta$  and  $\delta^*$  by approximately the same amount, down to no more than a factor of about ten.

The momentum thickness calculated from Eq. (25) is only slightly smaller than  $\delta^*$ , as would be expected from the profile flattening effect shown in Fig. 5. Figure 7 shows values of the so-called "shape factor"  $H = \delta^*/\theta$  for various Reynolds numbers and curvatures. The shape factor drops to nearly unity as  $(x/d)$  increases, indicating a nearly flat (slug flow) profile. There is substantial effect even at the very small curvature of  $x/d = 10$ . The somewhat erratic curve for  $x/d = 10,000$  reflects that in this case the viscous sub-layer extends nearly across the entire boundary layer. Except for a few cases near  $(x/d = 0)$ , we may state that curvature decreases  $\theta$  just as it does  $\delta^*$ . This seems to be in conflict with the accompanying drag increase until we remember that  $\theta$ , as defined by Eq. (25), is not a measure of drag, whereas in two-dimensional flow the flat plate drag coefficient  $C_D = 2\theta/L$ .

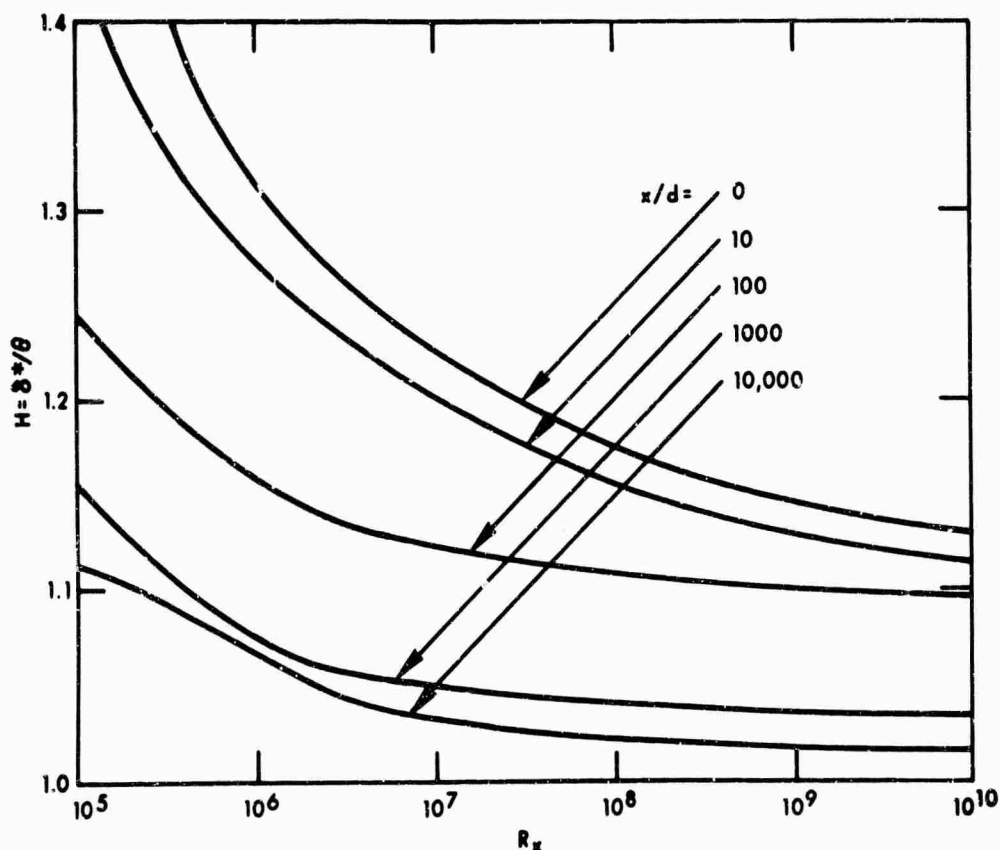


Fig. 7. Theoretical Values of the Turbulent Flow Shape Factor on Extremely Long Cylinders, from Eqs. (24) and (25)

Based upon his preliminary data, Richmond<sup>10</sup> suggested that  $\theta$ , as defined by Eq. (25), would correlate the local skin friction on a cylinder without extra parameters. He proposed that

$$C_{f(\text{cylinder})}/C_{f(\text{flat plate})} = \text{fcn}(\theta/d), \quad (26)$$

if the Reynolds number  $R_x$  is the same for both bodies. That this is not the case is shown in Fig. 8, where we see that a second variable,  $a^+$ , is also important. Richmond's four data points for very long cylinders are included

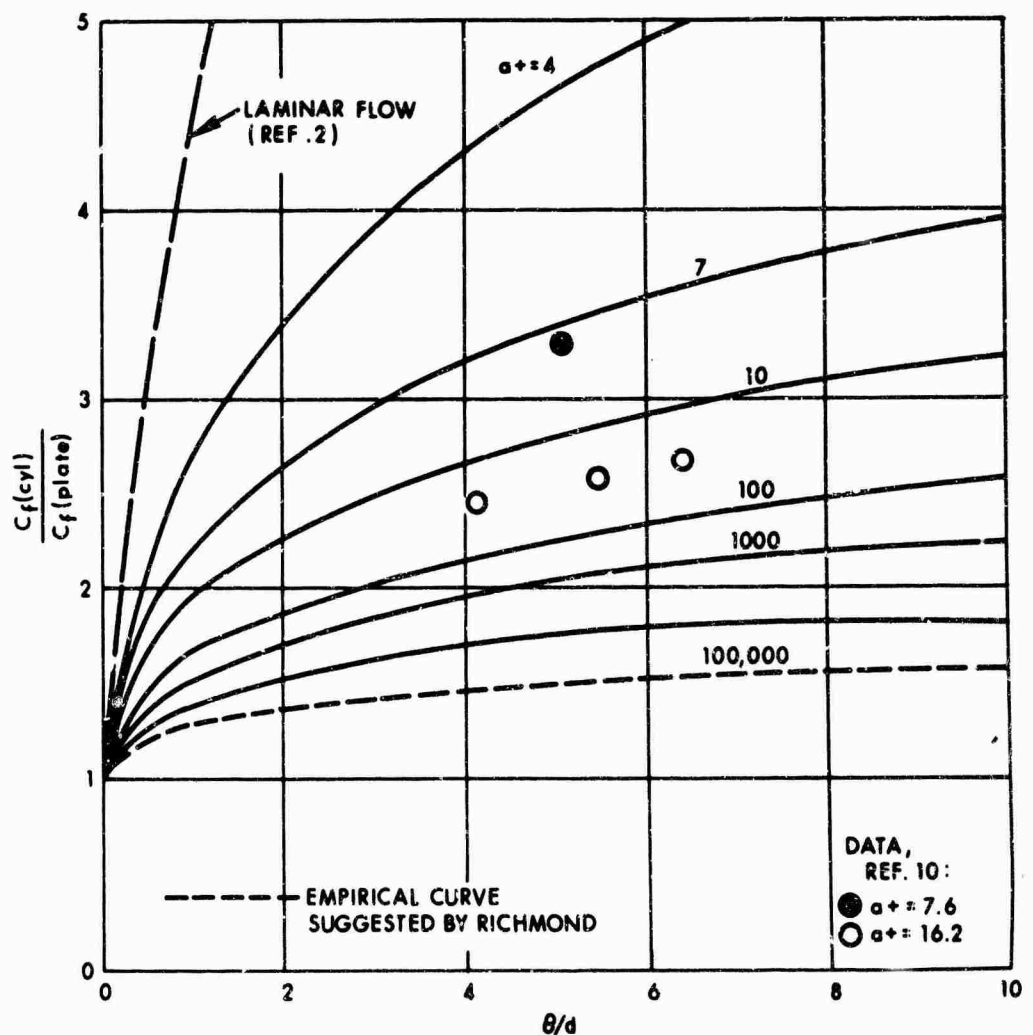


Fig. 8. Cylinder Skin Friction as a Function of Momentum Thickness and Curvature ( $a^+$ ), from Eqs. (17) and (25), Compared with the Data of Richmond (Ref. 10)

for comparison and are seen to be in excellent agreement with the present theory. The dotted curve shows Richmond's original suggested correlation, which is now clearly inadequate in view of the data reevaluation seen in the light of the modified law-of-the-wall. Since Fig. 8 requires two parameters to specify a point, it is now no better in principle than the presently suggested Fig. 3. In practice, Fig. 8 is much less handy than Fig. 3, because the calculation of  $\theta$  from Eq. (25) is a cumbersome matter, whereas the estimation of  $R_x$  and  $x/\delta$  is relatively simple.

### CONCLUSIONS

It has been shown that the modified cylindrical law-of-the-wall, Fig. 2, leads to a complete and rational analysis of the turbulent boundary layer in axial flow past long cylinders. The present theory is straightforward, leading to Eq. (17), and the numerical results are in excellent agreement with available experimental data. The theory seems to have no apparent limitations on cylinder size or Reynolds number, as long as the flow is turbulent. At low Reynolds numbers and large curvatures, the velocity profiles are mostly viscous sublayer, suggesting that in such cases the flow might actually be laminar. Tables and charts of important boundary layer parameters are easily constructed from the theory, and several charts are included here for the reader's interest. Also, the approach could easily be extended to pressure gradient flows if warranted.

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13. ABSTRACT <p>An analysis is presented which predicts the properties of an arbitrarily thick turbulent boundary layer in axial flow past a long cylinder. The study makes use of a modified form of the turbulent law-of-the-wall, which properly accounts for transverse curvature effects. Using this law, the theory which follows is then an exact solution to the axisymmetric equations of continuity and momentum in incompressible flow.</p> <p>Numerical results are given to show the effect of curvature on the various boundary layer characteristics. Skin friction and drag coefficients can be increased greatly with increasing curvature while boundary layer thickness is decreased. When defined in their axisymmetric form, the displacement and momentum thickness are both decreased by curvature. The velocity profile is flattened greatly and the shape factor <math>H = \delta^*/\theta</math> approaches unity at large curvature. The failure of earlier power-law theories to make accurate predictions is shown to be due to their inadequate handling of the strong profile shape changes. Finally, the earlier concept that the curvature effect on skin friction could be correlated by the ratio <math>(\theta/d)</math> is shown to be invalid.</p>			

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